# Weighted competition scale-free network

Shijun Wang* and Changshui Zhang
*State Key Laboratory of Intelligent Technology and Systems, Department of Automation, Tsinghua University, Beijing 100084, China*
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While many scale-free (SF) networks have been introduced recently for complex systems, most of them are binary random graphs and the rate at which the node in the network increases its connectivity depends on the time it arrived. We propose a model of weighted scale-free networks incorporating a fit-gets-richer scheme which means the connectivity of the node depends on both the degree and fitness of the node. The topology and weights of links of the network evolve as time goes on. The combined numerical and analytical approach indicates that asymptotically the scaling behaviors of the total weight distribution and the connectivity distribution are identical. The asymptotical sameness has also been observed in real networks.

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## I. INTRODUCTION

Many systems in the real world can be modeled as complex networks. Examples of complex networks include the Internet, science collaboration network, neural networks, metabolic networks, food web, etc. [1]. Recent research on complex networks has revealed a number of distinctive statistical properties that most networks seem to share [1–3]. One of the important measures of the topological structure of a network is its connectivity (also called degree) distribution $P(k)$, which is defined as the probability that a randomly selected node has exactly $k$ edges.

In traditional random graphs as well as in the small-world networks the connectivity distribution shows exponential decay in the tail. However, empirical studies on many real networks showed that the connectivity distribution exhibits a power-law behavior $P(k) \sim k^{-\gamma}$ for large $k$. Networks with power-law connectivity distributions are called scale-free (SF) networks. The first model of SF networks was proposed by Barabási and Albert (BA) [4].

In the BA model, networks grow at a constant rate and all nodes increase their connectivity in time as $k_i(t)=(t/t_i^0)^{\beta}$, where $\beta=1/2$ and $t_i^0$ is the time at which node $i$ was added into the system. In this way, the oldest nodes will have the highest number of links and a rich-gets-richer phenomenon appears. On the other hand, the nodes' connectivity and growth in real networks does not depend on their ages alone. For example, in the movie actor collaboration network, some famous movie stars acquire a very large number of links with other actors in a short time frame, much in excess of the majority of their contemporary or even older movie actors. On the WWW some webpages can acquire a large number of links through a combination of good content and marketing in a very short time, easily overtaking websites that have been around for much longer time. A phenomenon called fit-gets-richer appears in many real networks. To incorporate the different ability of the nodes to compete for links we assign a fitness parameter $\eta$ to each node accounting for the talent of an movie actor, the differences in the content of webpages [5–8].

*Electronic address: wsj02@mails.tsinghua.edu.cn

In most growing network models, all the links are considered equivalent. However, many real systems display different interaction strengths between nodes. Therefore, weighted growing network models are better models for real networks. Yook, Jeong, and Barabási (YJB) [9] took the first step in the direction of a systematic study of evolving networks with nonbinary connectivities. Hereafter, a class of models of weighted growing networks was proposed [10,11].

Zheng *et al.* [12] proposed a model of weighted scale-free networks incorporating a stochastic scheme for weight assignments to the links, taking into account both the popularity and fitness of a node based on the model of YJB. As the network grows the weights of links are driven either by the connectivity with probability $p$ or by the fitness with probability $1-p$. In the model of Zheng *et al.*, the weights of the links are changeless which is incompatible with many real networks. For example, in the movie actor collaboration network, the weight of the link (edge) between two actors increases every time when they act in a new movie.

To incorporate the fit-gets-richer and nonbinary weight phenomena in many real systems with the BA model, we propose a model called weighted competition scale-free (WCSF) network model which imitates the construction process of many real networks. The analytical result and numerical results show that asymptotically the total weight distribution converges to the scaling behavior of the connectivity distribution. The phenomenon has also been observed in real networks.

The plan of the paper is as follows. In Sec. II, we present our model and derive an analytical expression between the total weight and the connectivity of node. In Sec. III, we present the simulation results of our model. Results from real networks are presented in Sec. IV.

## II. WEIGHTED COMPETITION SCALE-FREE MODEL

For simplicity, we only consider undirected network model in this paper. Directed weighted evolving network model is a topic of our further research.

### A. Evolving model

The topological structure of our model follows that of the BA model of SF networks. The proposed model is defined by the following scheme.

 

*Step 1*. Start from a small number $m_0$ of fully connected nodes. At each time step, pick a preferred probability $p \in [0,1]$ and, with a uniform distribution, randomly generate a real number $s \in [0,1]$.

*Step 2*. $N$-step (node step). If $s \leqslant p$, we add a new node with fitness $\eta_i$ to the network, where $\eta$ is sampled from the distribution $\rho(\eta)$. Each new node has $m$ links that are connected to the nodes already presented in the system. According to the preferential attachment rule, the probability $\Pi_i$ of an existing node $i$ being selected for connection depends on the total number of links $k_i$ that node $i$ carries and on the fitness $\eta_i$ of that node—i.e.,

$$\Pi_i = \frac{\eta_i k_i}{\sum_j \eta_j k_j}. \tag{1}$$

*Step 3*. $E$-step (edge step). If $s > p$, we add $m$ new links to the network without introducing a new node. The two nodes as ends of a new link are selected according to the rule specified by formula (1). The weight of the new link is 1. If there is already a link between the selected two nodes, then we add 1 to the weight of the link. So the weights of the links in the network increase as time goes on.

### B. Degree distribution

To obtain the degree distribution of the WCSF network model, we derive an analytical expression for the degree $k_i(\eta_i, t_i^0, t)$ of a node $i$ with fitness $\eta_i$ at time $t$ which is added to the network at time $t_i^0$. At $E$-step, if node $i$ is selected as one end of a new link, then the probability $P_{in}(i)$ that the node at the other end of the link is node $i$'s immediate neighbor (nodes connected directly with $i$) is

$$P_{in}(i) = \frac{\sum_n \eta_n k_n}{\sum_j \eta_j k_j}, \tag{2}$$

where node $n$ is node $i$'s immediate neighbor. So when we add a new link to the network, the probability $P_{din}(i)$ that the degree of node $i$ increases by 1 is

$$P_{din}(i) = \frac{\eta_i k_i}{\sum_j \eta_j k_j}\left(1 - \frac{\eta_i k_i}{\sum_j \eta_j k_j} - \frac{\sum_n \eta_n k_n}{\sum_j \eta_j k_j}\right). \tag{3}$$

The rate that node $i$ increases its connectivity at each time step is

$$\begin{aligned}\frac{\partial k_i}{\partial t} &= pm\frac{\eta_i k_i}{\sum_j \eta_j k_j} + (1-p)mP_{din}(i)\\ &= pm\frac{\eta_i k_i}{\sum_j \eta_j k_j} + (1-p)m\\ &\quad\times\frac{\eta_i k_i}{\sum_j \eta_j k_j}\left[1 - \frac{\eta_i k_i}{\sum_j \eta_j k_j} - \frac{\sum_n \eta_n k_n}{\sum_j \eta_j k_j}\right].\end{aligned} \tag{4}$$

We assume that $\rho(\eta)$ is a distribution in the interval $[\eta_{\min}, \eta_{\max}]$ and $\eta_{\min} > 0$. Under some distributions of $\rho(\eta)$, the Bose–Einstein condensation phenomenon may occur, which means the fittest node grabs a finite fraction of all links [14]. Assuming that the given distribution $\rho(\eta)$ will not lead to BE condensation, we can get

$$\frac{\eta_i k_i}{\sum_j \eta_j k_j} \xrightarrow{t\to\infty} 0 \tag{5}$$

and

$$\sum_n \eta_n k_n \leqslant \eta_{\max} \sum_n k_n. \tag{6}$$

The expectation of the sum of degrees of the immediate neighbors of node $i$ is

$$E\left(\sum_n k_n\right) = z_2, \tag{7}$$

where $z_2$ is the expectation of the number of $i$'s second neighbors (nodes connected with $i$'s immediate neighbors).

Assuming that $G_0(x)$ is the generating function for the probability distribution $P(k)$ of node degree [13], the degree of the nodes that we arrive at by following a randomly chosen link has a distribution which is generated by

$$\frac{\sum_k kp_k x^k}{\sum_k kp_k} = x\frac{G_0'(x)}{G_0'(1)}. \tag{8}$$

If we start at a randomly chosen node and follow each of the edges at that node to reach the $k$ nearest neighbors, then the nodes arrived at each have the distribution of remaining outgoing edges generated by Eq. (8), less one power of $x$, to allow for the link that we arrived along. Thus the distribution of outgoing edges is generated by the function

$$G_1(x) = \frac{G_0'(x)}{G_0'(1)} = \frac{1}{z}G_0'(x), \tag{9}$$

where $z$ is the average node degree. The probability that any of these outgoing edges connects to the original node that we started at, or to any of its other immediate neighbors, goes as $1/N$ and hence can be neglected in the limit of large $N$.

Then the expectation of the number of the second neighbors of node $i$ is [13]

$$z_2 = \left[\frac{d}{dx}G_0(G_1(x))\right]_{x=1} = G_0'(1)G_1'(1) = G_0''(1). \tag{10}$$

For a given generating function $G_0(x)$, $z_2$ is a constant. So when the network has evolved for a long time,

$$\frac{\sum_n \eta_n k_n}{\sum_j \eta_j k_j} \approx \frac{E\left(\sum_n \eta_n k_n\right)}{\sum_j \eta_j k_j} \leqslant \frac{\eta_{\max} z_2}{\sum_j \eta_j k_j} \xrightarrow{t\to\infty} 0. \tag{11}$$

The rate that node $i$ increases its connectivity at each time step is approximate:

$$\begin{aligned}\frac{\partial k_i}{\partial t} &= pm\frac{\eta_i k_i}{\sum_j \eta_j k_j} + (1-p)m \\ &\quad \times \frac{\eta_i k_i}{\sum_j \eta_j k_j}\left[1 - \frac{\eta_i k_i}{\sum_j \eta_j k_j} - \frac{\sum_n \eta_n k_n}{\sum_j \eta_j k_j}\right] \\ &\approx pm\frac{\eta_i k_i}{\sum_j \eta_j k_j} + (1-p)m\frac{\eta_i k_i}{\sum_j \eta_j k_j} \\ &= m\frac{\eta_i k_i}{\sum_j \eta_j k_j}. \end{aligned} \tag{12}$$

If $\rho(\eta)=\delta(\eta-1)$—i.e., all fitness are equal—Eq. (12) reduces to the scale-free model which predicts that $k_i(t)\sim t^{1/2}$ [4].

The mean-field solution for Eq. (12) [5,15] is

$$k_i(\eta_i, t_i^0, t) = m\left(\frac{t}{t_i^0}\right)^{\beta(\eta_i)}, \tag{13}$$

where $\beta(\eta_i)=\eta_i/C$, $C=\int\rho(\eta)\{\eta/[1-\beta(\eta)]\}d\eta$, $\langle\Sigma_j \eta_j k_j\rangle \xrightarrow{t\to\infty} Cmt$. If $\rho(\eta)$ is chosen uniformly from the interval $(0,1]$, $C=1.255$ [5].

### C. Total weight distribution

In binary networks the node's importance is characterized by the total number of the links it has. Similarly, in a weighted network the importance of a node can be measured by its total weight, obtained by summing up the weights of the links that connect to it, $w_i=\Sigma_j w_{ij}$. $w_i$ increases in both $N$-step and $E$-step in the WCSF model:

$$w_i(\eta_i, t_i^0, t) = m + \sum_{t'=t_i^0}^{t}\{pmP_i^N(t') + (1-p)mP_i^E(t')\}, \tag{14}$$

where $P_i^N(t')$ is the probability that node $i$ is selected to be connected to a new node at time $t'$ in $N$-step and $P_i^E(t')$ is the probability that node $i$ is selected to be connected to an old node at time $t'$ in $E$-step.

According to the preferential attachment rule specified by formula (1), $P_i^N(t')=P_i^E(t')$. Assuming that $t_i^0$ is far from $t^0$ which is start time of the network,

$$\begin{aligned} w_i(\eta_i, t_i^0, t) &= m + \sum_{t'=t_i^0}^{t}\{pmP_i^N(t') + (1-p)mP_i^E(t')\} \\ &= m + \sum_{t'=t_i^0}^{t}\{mP_i^N(t')\} \\ &= m + m\sum_{t'=t_i^0}^{t}\left\{\frac{\eta_i k_i}{\sum_j \eta_j k_j}\right\} \\ &\approx m + m\sum_{t'=t_i^0}^{t}\left\{\frac{\eta_i m\left(\frac{t'}{t_i^0}\right)^{\eta_i/C}}{Cmt'}\right\} \\ &\approx m + \frac{m\eta_i}{C}\int_{t_i^0}^{t}\frac{\left(\frac{t'}{t_i^0}\right)^{\eta_i/C}}{t'}dt' \\ &= m\left(\frac{t}{t_i^0}\right)^{\eta_i/C} \\ &= k_i(\eta_i, t_i^0, t), \end{aligned} \tag{15}$$

$w_i(\eta_i, t_i^0, t)\approx k_i(\eta_i, t_i^0, t)$, which means that the degree distribution $P(k)$ and the total weight distribution $P(w)$ are approximately identical.

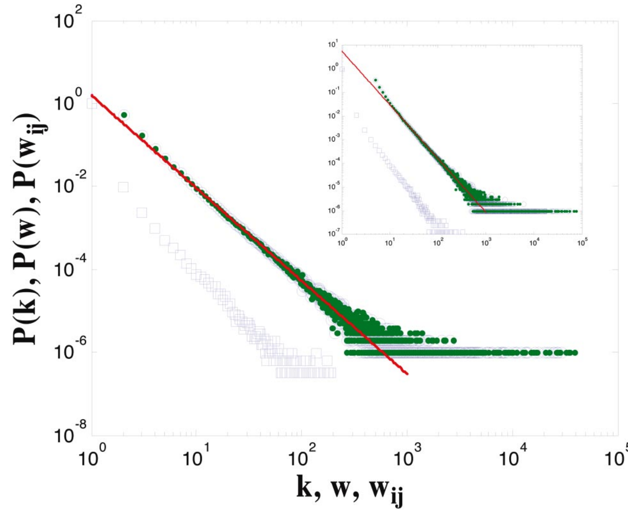


FIG. 1. (Color online) Distribution $P(k)$ (○) of connectivity and distribution $P(w)$ (●) of total weight for the WCSF model for $m=m_0=2$ and $p=0.6$. The slope $\gamma$ of the power-law distribution for $P(k)$ and $P(w)$ is −2.2378 and −2.2427. The squares (□) are the distribution $P(w_{ij})$ of individual link weights with $\gamma=-2.8890$. The line is a guide line with slop −2.2378. Inset: distribution $P(k)$ (○) of connectivity and distribution $P(w)$ (●) of total weight for the WCSF model for $m=m_0=5$ and $p=0.6$. The slope $\gamma$ of the power-law distribution for $P(k)$ and $P(w)$ is −2.2685 and −2.2697.

## III. SIMULATION RESULTS

We performed extensive numerical simulations on the WCSF model. In our simulations, we studied networks up to $N=1\times 10^6$ nodes with $m=m_0=2$ and $m=m_0=5$.

Figure 1 shows the distribution of $P(k)$, $P(w)$ and the distribution $P(w_{ij})$ of weights of links. From Fig. 1 we can see that the distribution $P(w_{ij})$ of the individual link weights also follows a power-law distribution which is consistent with real networks. It's clear that there is little difference between $P(k)$ and $P(w)$. Again, the approximate equality of two distributions is reflected in the dynamical behavior of $k_i(t)$ and $w_i(t)$, as Fig. 2 indicates.

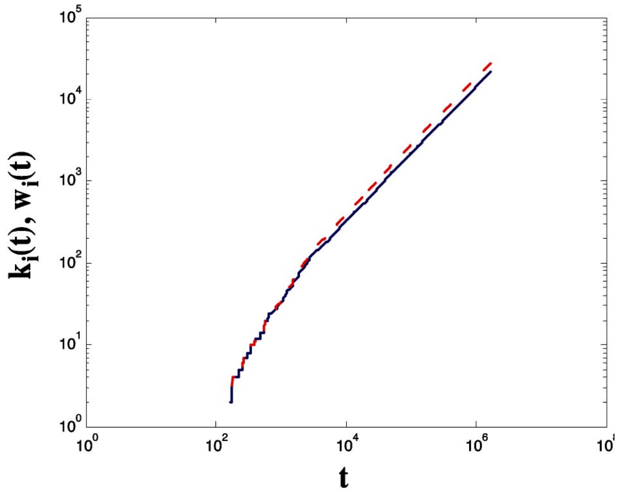


FIG. 2. (Color online) $k_i(t)$ (solid line) and $w_i(t)$ (dotted line) vs $t$ for a randomly selected node $i$ for the WCSF model ($i=100$).

## IV. RESULTS OF REAL NETWORKS

In order to show the validity of our model, we calculate the degree distributions and total weight distributions of book borrowing network and movie actor collaboration network.

### A. Book borrowing network

In the book borrowing network (BBN), the nodes are the books, and two nodes have a common link if the corresponding books have been borrowed together by a person at a time. If two books have been borrowed together more than once, then the weight of the link between them is accumulated. The popularity of a book can be seen as a competition factor which attracts people to borrow it. So the book borrowing network is a weighted competition network.

We first analyze the data set under consideration. Our experiments draw on data collected from Library of Tsinghua University, China. Figure 3 is an example of Tsinghua University Library's book borrowing records. The first column is ID's of books and the third is ID's of readers. The borrowing time is listed in the middle column. The data set consists of book borrowing records from 15:35:00 March 4, 2003 through 11:21:00 November 12, 2003. In this period there are 573 862 book borrowing records in which 142 081 books appear.

Figure 4 shows the distribution of $P(k)$ and $P(w)$ for the book borrowing network. It is clear that the distribution of degree and distribution of total weight are approximately identical.

| | | |
|---|---|---|
| b1010768x; | 03-04-08 15:35; | p10583956 |
| b10107939; | 03-04-08 10:18; | p10824522 |
| b10126235; | 03-04-08 09:28; | p10809545 |
| b10255977; | 03-04-08 14:05; | p10633716 |

FIG. 3. Book borrowing records.

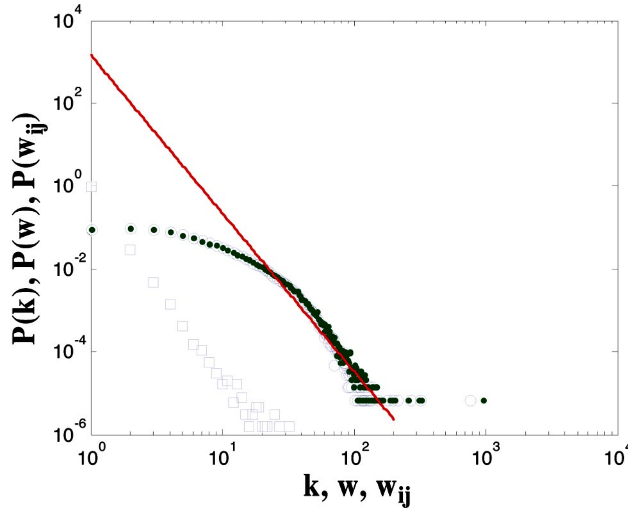


FIG. 4. (Color online) Distribution $P(k)$ (○) of connectivity and distribution $P(w)$ (●) of total weight for the book borrowing network. The slope $\gamma$ of the power-law distribution for $P(k)$ and $P(w)$ is −3.8191 and −3.8471. The squares (□) are the distribution $P(w_{ij})$ of individual link weights with $\gamma=-4.2543$. The line is a guide line with slop −3.8191 ($k$ is the number of books been borrowed together, $w_{ij}$ the times that two books have been borrowed together).

### B. Movie actor collaboration network

A much-studied database is the movie actor collaboration network (MACN), based on the Internet Movie Database, which contains all movies and their casts since the 1890s. In this network the nodes are the actors, and two nodes have a common link if the corresponding actors have acted in a movie together. If two actors appear together in $N$ movies, then the weight of the link between these two actors is $N$. Also, in Hollywood some actors in a very short time frame build a movie portfolio and a collection of links that easily surpasses many actors in business for much longer time. So the movie actor collaboration network is also a weighted competition network.

Figure 5 shows the distribution of $P(k)$ and $P(w)$ for the movie actor collaboration network [20]. The analytical result is supported by the real network again.

### C. Linear dependence of the total weight of the nodes with the connectivity

Some work on weighted network show a nonlinear dependence of the total weight of the nodes with the connectivity. Barrat *et al.* [16] found that the average total weight $w(k)$ of nodes with degree $k$ increases with the degree as $w(k)\sim k^{\beta}$. In the absence of correlations between the weight of edges and the degree of nodes, $w(k)=\langle w_{ij}\rangle k$, where $\langle w_{ij}\rangle$ is the average weight in the network. $\beta>1$ implies that the weight of edges belonging to highly connected nodes tends to have a value higher than that of edges belonging to nodes with lower degree. This denotes a strong correlation between the weight and topological properties in the network. For example, in the world airport network [17–19], the larger is an airport, the more traffic it can handle.

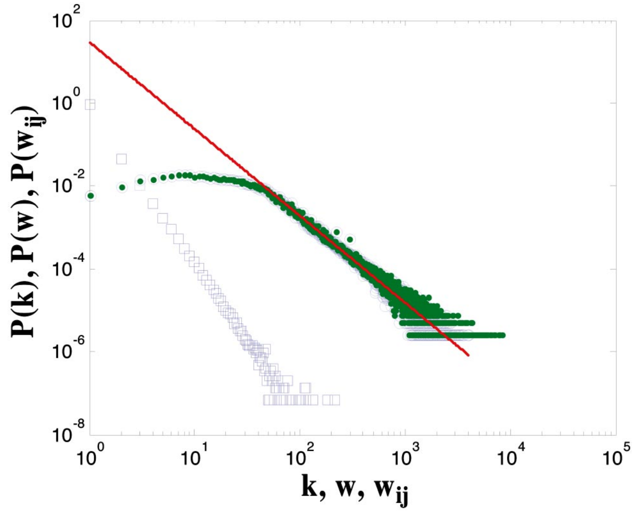


FIG. 5. (Color online) Distribution $P(k)$ (○) of connectivity and distribution $P(w)$ (●) of total weight for the movie actor collaboration network. The slope $\gamma$ of the power-law distribution for $P(k)$ and $P(w)$ is −2.0958 and −2.0552. The squares (□) are the distribution $P(w_{ij})$ of individual link weights with $\gamma=-3.7758$. The line is a guide line with slop −2.0958. ($k$ is the number of collaborators, $w_{ij}$ the times that two actors have acted together.)

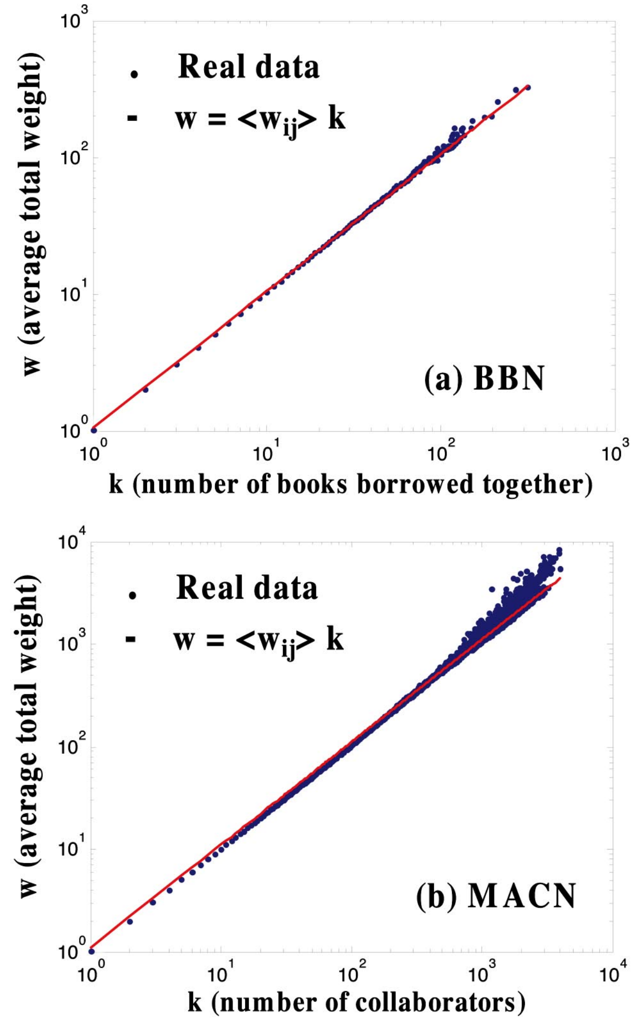


FIG. 6. (Color online) Average total weight $w(k)$ as function of the degree $k$ of nodes. (a) Book borrowing network. (b) Movie actor collaboration network.

In Fig. 6, we show the average total weight as function of the degree $k$ of nodes in the BBN and MACN. For the BBN and MACN, the curves are very similar and well fitted by the uncorrelated approximation $w(k)=\langle w_{ij}\rangle k$. The total weight of a node is then simply proportional to its degree, yielding an exponent $\beta=1$, and the two quantities provide therefore the same information on the system.

## V. CONCLUSION

In summary, we proposed and studied a model of weighted competition scale-free networks in which the weights assigned to links determined by the connectivity and fitness of nodes as the network grows increase gradually. The total weight distribution of the WCSF model follows a power-law probability distribution and is approximately the same as the degree distribution. The model also leads to a power-law distribution for the weights of individual links. An expression relating the total weight and the connectivity of nodes was derived analytically. The analytical result was supported by the numerical results. The same feature was also found in the real networks.

The results presented in this paper represent only the starting point toward understanding weighted competition networks. Under some distributions of fitness, the Bose–Einstein condensation phenomenon may occur with one node grabbing a finite fraction of all the links [14]. The research about the thermodynamically distinct phases of the weighted competition scale-free networks is a real challenge for the future research.

## ACKNOWLEDGMENT

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